

Imaginary mass lune determinants

J.S.Dowker¹

*Theory Group,
School of Physics and Astronomy,
The University of Manchester,
Manchester, England*

Functional determinants for a scalar field with negative mass squared are numerically evaluated on an orbifolded three-sphere, in particular on a lune and on a regular 4-polytope fundamental domain. Graphs are provided of the logdets and some Hartle–Hawking probabilities, on the basis of the dS/CFT correspondence.

¹dowker@man.ac.uk , dowkeruk@yahoo.co.uk

1. Introduction.

This is a further installment of my ongoing numerical evaluations of functional determinants in various domains, mostly spherical.

In a previous communication, [1], I computed the determinant for a scalar field with imaginary mass, $i\alpha$, on a lens space *i.e.* a \mathbb{Z}_q factor of the three-sphere without fixed points. In this parallel work, I repeat the calculation for such an action but now one *with* fixed points. It is not anticipated that the results will differ greatly, ultimately, from those for the whole sphere. Nevertheless there is, I believe, some technical merit in the intervening calculation and I present it for this reason.

I spend most time on the simplest orbifold, the periodic lune, and then treat, in a later section, the more general quotient. Some numbers for periodic spherical tetrahedra are exhibited for potential amusement.

2. Lunes

The same eigenstructure holds for lunes as for lens spaces in that the zeta function takes the same form as in [1], only the degeneracies change. In fact one can proceed indirectly to the generating function by making this form more precise, as in [2] for the full sphere. The relevant ζ -function is that employed in [2] which I repeat here for the three-lune specifically (see also [3] for a more extended description of the geometric situation),

$$\zeta(s, a, \alpha, q) \equiv \sum_{m_1, m_2, m_3=0}^{\infty} \frac{1}{((a + q m_1 + m_2 + m_3)^2 - \alpha^2)^s}, \quad (1)$$

This is actually the ζ -function for a lune of *any* angle π/q . When q is an integer one has a fixed point \mathbb{Z}_q action and I restrict to this case from now on. The lune has Neumann conditions on the (spherical) sides if $a = a_N = 1$ and Dirichlet if $a = a_D = 1 + q$. I will, shortly, add these two cases to give the eigenstructure of the closed *periodic lune*, which is the round (whole) sphere when $q = 1$.

The determination of the degeneracies is a combinatorial, lattice problem but one really needs only their generating function. I define the integer, l , by,

$$l \equiv q m_1 + m_2 + m_3, \quad (2)$$

which now runs from 0 upwards. The degeneracy, $g(l, q)$, of the *level* $\omega(l) \equiv a + l$ for a given l is the number of (nonnegative) integers m_1, m_2, m_3 such that the equality

(2) holds. The associated generating function is classic, going back at least to Euler, and is,

$$h(t, q) = \sum_{l=0}^{\infty} g(l, q) t^l = \frac{1}{(1-t)^2 (1-t^q)}. \quad (3)$$

I now combine the Neumann and Dirichlet problems on the π/q lune. This gives the spectrum on the *periodic* $2\pi/q$ lune and I introduce the corresponding total ζ -function,

$$\zeta(s, \alpha, q) \equiv \zeta(s, a_N, \alpha, q) + \zeta(s, a_D, \alpha, q). \quad (4)$$

Both sets of eigenlevels are to be labelled by the same integer l with $l = 0, 1, 2, \dots \infty$. Because the constant $a_D = a_N + q$, the Dirichlet generating function is the Neumann one with l shifted by q , and the total generating function is again standard, being,

$$H(t, q) = \sum_{l=0}^{\infty} G(l, q) t^l = \frac{1+t^q}{(1-t)^2 (1-t^q)}. \quad (5)$$

which can be used to compute any individual degeneracy.

The ζ -function, (4), can therefore be written in general form exactly as in [1],

$$\begin{aligned} \zeta(s, \alpha, q) &= \sum_{l=0}^{\infty} \frac{G(l, q)}{((1+l)^2 - \alpha^2)^s} \\ &\equiv \sum_{l=0}^{N-1} \frac{G(l, q)}{((1+l)^2 - \alpha^2)^s} + \zeta^{(N)}(s, \alpha, q), \end{aligned} \quad (6)$$

and one easily gets,

$$\zeta'(0, \alpha, q) = \zeta^{(N)'}(0, \alpha, q) - \sum_{l=0}^{N-1} G(l, q) \log |((1+l)^2 - \alpha^2)| + i\pi \sum_{l=0}^{[\alpha]-1} G(l, q). \quad (7)$$

(Remember, a is integral.)

Again I have extracted N levels, achieved by removing the first N terms in the Taylor expansion of the generating function (5). I express this subtraction by $H^{(N)}(t, q)$.

3. Computing the derivative at zero

The same procedure of back integrating a convergent resolvent, employed previously, will be followed. Again the reference point is set to $\alpha = 0$ because this ζ -function has already been studied and computational formulae are available, *e.g.* [4]. For completeness here, I will later repeat these results.

Although the equations closely parallel those detailed elsewhere, I give the intermediate steps because I wish to derive the final formula more generally and also to polish the formalism slightly.

Differentiating (6) twice with respect α^2 then taking the derivative at $s = 0$ and finally back integrating with respect to α^2 gives,

$$\zeta^{(N)'}(0, \alpha, q) = \int_0^{\alpha^2} dw \int_0^w dw' \sum_{l=N}^{\infty} \frac{G(l, q)}{((1+l)^2 - w')^2} + C_N(\alpha, q) \quad (8)$$

where the last term arises from the constants of integration and involves $\zeta(s, \alpha, q)$ at $\alpha = 0$. (Any lower limits could be chosen provided the value of α is in the easily computed range $0 \leq \alpha \leq 1/2$. Another good choice would be the conformal one, $\alpha = 1/2$.)

The first constant of integration requires the computation of

$$\left. \frac{d}{d\alpha^2} \zeta^{(N)'}(0, \alpha, q) \right|_{\alpha=0} = \left. \frac{d}{d\alpha^2} \zeta'(0, \alpha, q) \right|_{\alpha=0} + \sum_{l=0}^{N-1} G(l, q) \log(1+l)^2$$

using (6). Now

$$\frac{d}{d\alpha^2} \zeta(s, \alpha, q) = s \zeta(s+1, \alpha, q). \quad (9)$$

and, noting that $\zeta(s, \alpha, q)$ does not have a pole at $s = 1$, then,

$$\left. \frac{d}{d\alpha^2} \zeta'(0, \alpha, q) \right|_{\alpha=0} = \zeta(1, 0, q). \quad (10)$$

The absence of a pole is true for all odd-dimensional, closed manifolds and is, perhaps, more familiar here as the vanishing of the $C_{1/2}$ heat-kernel coefficient. Even if there were a pole, the right-hand side of (10) would still be finite, equal to the pole remainder. This could be made the basis of a limited computational method that avoids differentiating twice.

Therefore

$$C_N(\alpha, q) = \alpha^2 (\zeta(1, 0, q) - A_N(q)) + \zeta'(0, 0, q) + B_N(q)$$

where,

$$B_N(q) = \sum_{l=0}^{N-1} G(l, q) \log(1+l)^2,$$

and

$$A_N(q) = \sum_{l=0}^{N-1} \frac{G(l, q)}{(1+l)^2}.$$

For the integral in (8), a Bessel Laplace transform gives,

$$\sum_{l=N}^{\infty} \frac{G(l, q)}{((1+l)^2 - \alpha^2)^2} = \sqrt{\pi} \int_0^{\infty} d\tau e^{-\tau} H^{(N)}(e^{-\tau}, q) \left(\frac{\tau}{2\alpha}\right)^{3/2} I_{3/2}(\alpha\tau). \quad (11)$$

The quantity $e^{-\tau} H^{(N)}$ is the (subtracted) cylinder kernel for the ‘square-root’ pseudo operator whose eigenvalues are $(1+l)$.

The double integration with respect to α^2 is best effected using Bessel properties, *e.g.* Petiau, [5], chap.III §§12–15. Set $z = \alpha\tau$ and use the easily derived relation,

$$\int_0^z z_1 dz_1 \dots \int_0^{z_{p-1}} z_p dz_p z_p^{-\lambda} I_{\lambda}(z_p) = z^{p-\lambda} I_{\lambda-p}^{(p)}(z),$$

where $F^{(p)}$ stands for the removal of the first p terms in the power series expansion of F . In d dimensions, $p = d - 1$ and $\lambda = d/2$.

The value of λ relevant here is $3/2$ for which, if needed ($p = 2$),

$$I_{-1/2}(z) = \sqrt{\frac{2}{\pi z}} \cosh z.$$

Hence putting everything together, the final, computable formula from (7) and (8) is (I give the real part only),

$$\begin{aligned} \zeta'(0, \alpha, q) &= \sqrt{2\pi\alpha} \int_0^{\infty} d\tau \frac{1}{\sqrt{\tau}} e^{-\tau} H^{(N)}(e^{-\tau}, q) I_{-1/2}^{(2)}(\alpha\tau) \\ &+ \sum_{l=0}^{N-1} G(l, q) \left(\log \frac{(l+1)^2}{|\alpha^2 - (l+1)^2|} - \alpha^2 \frac{1}{(l+1)^2} \right) \\ &+ \alpha^2 \zeta(1, 0, q) + \zeta'(0, 0, q). \end{aligned} \quad (12)$$

The last two terms have to be found independently. I give the computable expressions derived from those in [4] by setting α equal to zero,

$$\begin{aligned} \zeta(1, 0, q) &= \frac{1}{4} \int_0^{\infty} dx \operatorname{Re} \frac{\tau \coth q\tau/2}{\sinh^2 \tau/2} \\ \zeta'(0, 0, q) &= \frac{1}{2} \int_0^{\infty} dx \operatorname{Re} \frac{\coth q\tau/2}{\tau \sinh^2 \tau/2}, \end{aligned} \quad (13)$$

with $\tau = x + iy$, y having to lie between 0 and the first singularity of the integrand above the real axis.

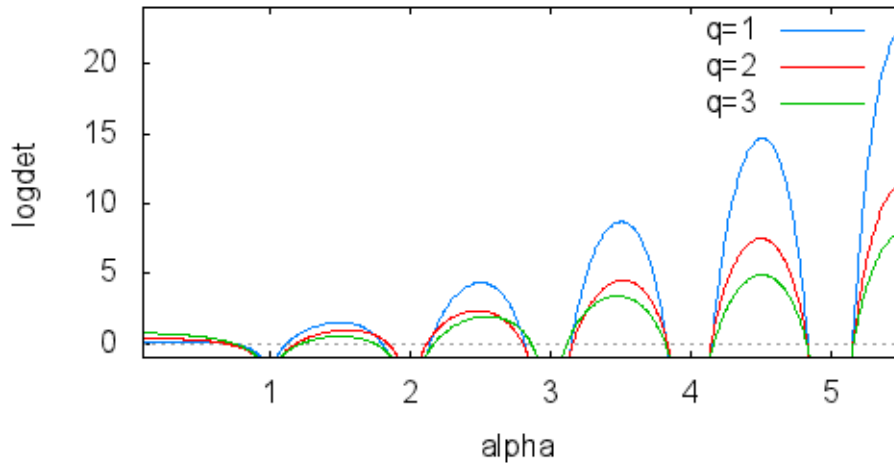
As N tends to infinity, the integral term in (7) vanishes leaving a standard regularised expression equivalent to that used in, *e.g.*, [6] for the round sphere. In fact, for largish N , one could drop the integral, with a reduced accuracy penalty.

Incidentally, an equation similar to (12) will be valid for a more general spectrum because the Bessel transform separates λ from the mass parameter, α^2 , in the eigenvalues $\lambda - \alpha^2$, [7]. Whether all quantities in (12) are available depends on the system.

4. The results

Figure 1 shows the variation of the logdet for the three lunes, $q = 1$ (the round sphere), $q = 2$ and $q = 3$ only, the round sphere being typical for these low values of q . Figure 2 enlarges the region $0 < \alpha < 1$ and demonstrates that, as for lens spaces, any deformation from roundness eliminates the maximum at $\alpha = 1/2$, the conformal point. Figure 3 plots the Hartle–Hawking probability, $|\Psi_{HH}|^2$, and Figure 4 displays the same numbers plotted against the variable $\sigma = 1/4 - \alpha^2$, used in [6,8]. Like these references I have, without loss of generality, set $N = 2$ in the free $\text{Sp}(N)$ CFT₃ of anticommuting scalars according to the dS/CFT scheme.

Fig.1. q-Lune logdet. Imaginary mass



If one wishes to plot the wavefunction(al) itself, then the imaginary part in (7) comes into play. Since the degeneracy, $G(l, q)$, alternates between even and odd as l increases, the zeros of the wavefunction will alternate between points of inflection

and extrema, which are maxima and minima, in turn.

Figure 5 displays the results for three higher values of q which have a more complicated initial behaviour.

Fig.2. q -Lune logdet. Imaginary mass

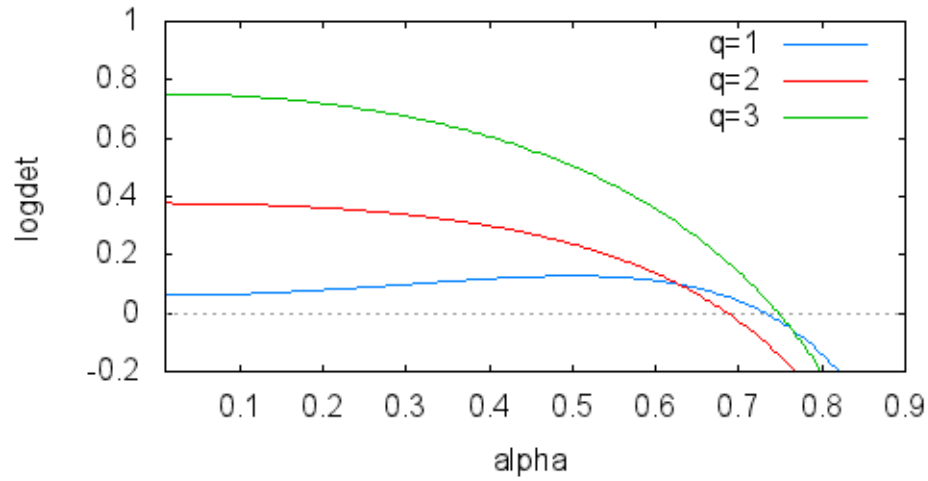


Fig.3. q -Lune Hartle-Hawking probability

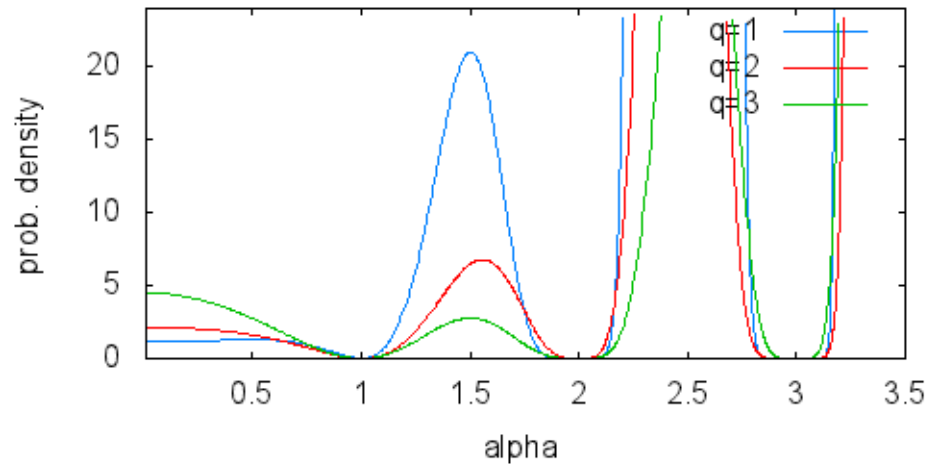


Fig.4. q-Lune Hartle-Hawking probability

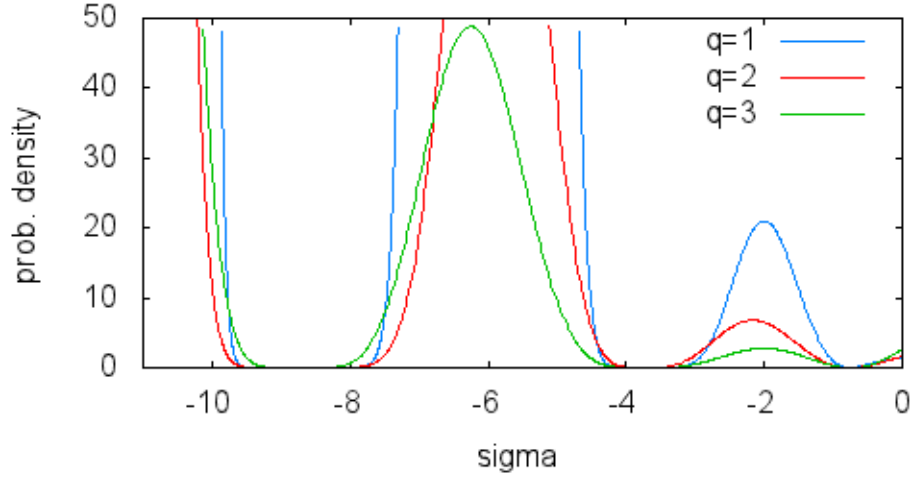
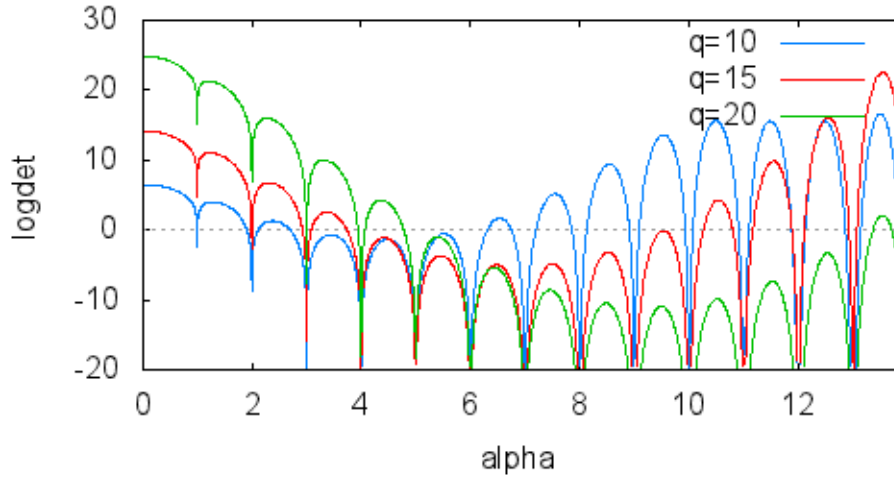


Fig.5. q-Lune logdet. Imaginary mass



5. Other orbifold quotients

The lune is the simplest orbifold quotient of the three-sphere and its analysis serves as an introduction to the general case. In fact the (allowed) factors of the three-sphere can all be treated in a unified way, as I now bring forward, if only for the sake of completing a list. The quotients are classified by the regular 4-polytope reflection groups, of which there are only a finite number.

The eigenvalues all have the same general form, (2) being replaced by,

$$l = \omega_1 n_1 + \omega_2 n_2 + \omega_3 n_3 ,$$

with the parameters, ω_i , taking particular integer values. For convenience I list these as vectors : (3, 4, 5) for the polytope group [3, 3, 3], (4, 6, 8) for [3, 3, 4], (6, 8, 12) for [3, 4, 3] and (12, 20, 30) for [3, 3, 5]. I have labelled the groups by their Coxeter symbol. (The Schläfli symbol of the corresponding 4-polytope is $\{\omega_1, \omega_2, \omega_3\}$.)

The quotients are fundamental domains for the action and tile the three-sphere. In the periodic case they can be pictured as (doubled) spherical tetrahedra, or limiting forms thereof.

The degeneracy generating functions are classic and follow from Molien's theorem and invariant theory. I give them in the guise of the cylinder kernel, $K^{1/2}$,

$$\begin{aligned} K^{1/2}(\tau, \boldsymbol{\omega}) &= 2 \cosh(\omega_0 \tau/2) \prod_{i=1}^3 \frac{1}{2 \sinh(\omega_i \tau/2)} \\ &= t \frac{1 + t^{\omega_0}}{(1 - t^{\omega_1})(1 - t^{\omega_2})(1 - t^{\omega_3})}, \end{aligned} \quad (14)$$

where ω_0 is the number of reflections and equals $1 + \sum_{i=1}^3 (\omega_i - 1)$. Note that $K^{1/2}(\tau, \boldsymbol{\omega})$ is antisymmetric in τ . (See [9].)

The quantity multiplying t in (14), is the generating function, $H(t, \boldsymbol{\omega})$, cf (5). The two terms in the numerator correspond to Neumann and Dirichlet conditions on the boundary of the fundamental domain. Adding these yields the spectrum on a *periodic* domain, obtained by sticking two fundamental domains together and identifying boundaries. This doubled domain has singularities of codimensions 2 and 3.

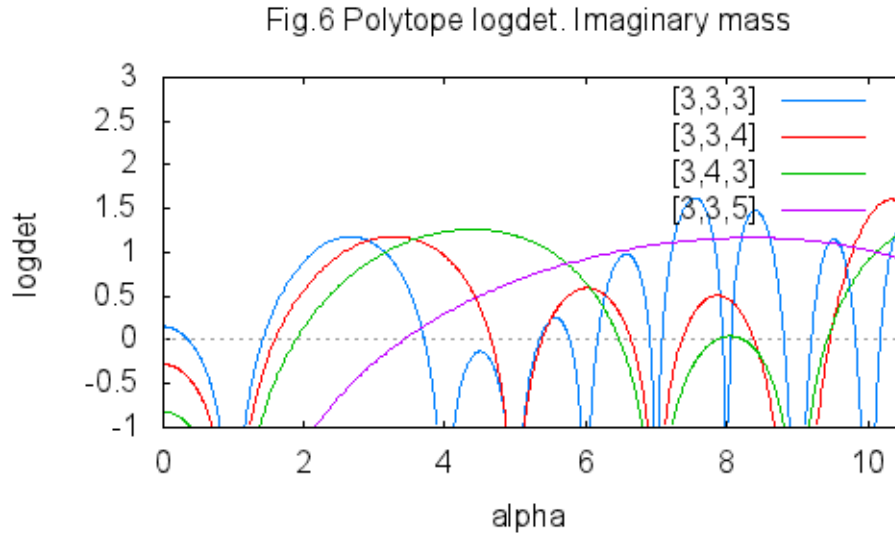
The analysis is virtually identical to that in the preceding sections and leads to the final formula,

$$\begin{aligned} \zeta'(0, \alpha, \boldsymbol{\omega}) &= \sqrt{2\pi\alpha} \int_0^\infty d\tau \frac{1}{\sqrt{\tau}} K^{1/2(N)}(\tau, \boldsymbol{\omega}) I_{-1/2}^{(2)}(\alpha\tau) \\ &\quad + \sum_{l=0}^{N-1} G(l, \boldsymbol{\omega}) \left(\log \frac{(l+1)^2}{|\alpha^2 - (l+1)^2|} - \alpha^2 \frac{1}{(l+1)^2} \right) \\ &\quad + \alpha^2 \zeta(1, 0, \boldsymbol{\omega}) + \zeta'(0, 0, \boldsymbol{\omega}). \end{aligned} \quad (15)$$

The equations which generalise (13) can be obtained, as above, from [4],

$$\begin{aligned} \zeta(1, 0, \boldsymbol{\omega}) &= \frac{1}{4} \int_0^\infty dx \operatorname{Re} \frac{\tau \cosh((\omega_1 + \omega_2 + \omega_3 - 2)\tau/2)}{\sinh(\omega_1 \tau/2) \sinh(\omega_2 \tau/2) \sinh(\omega_3 \tau/2)} \\ \zeta'(0, 0, \boldsymbol{\omega}) &= \frac{1}{2} \int_0^\infty dx \operatorname{Re} \frac{\cosh((\omega_1 + \omega_2 + \omega_3 - 2)\tau/2)}{\tau \sinh(\omega_1 \tau/2) \sinh(\omega_2 \tau/2) \sinh(\omega_3 \tau/2)}. \end{aligned} \quad (16)$$

Figure 6 gives some numerical results.



To complete the list, Figures 7 and 8 give the results for the (double) dihedral group for which $\omega = (q, 2, 1)$.

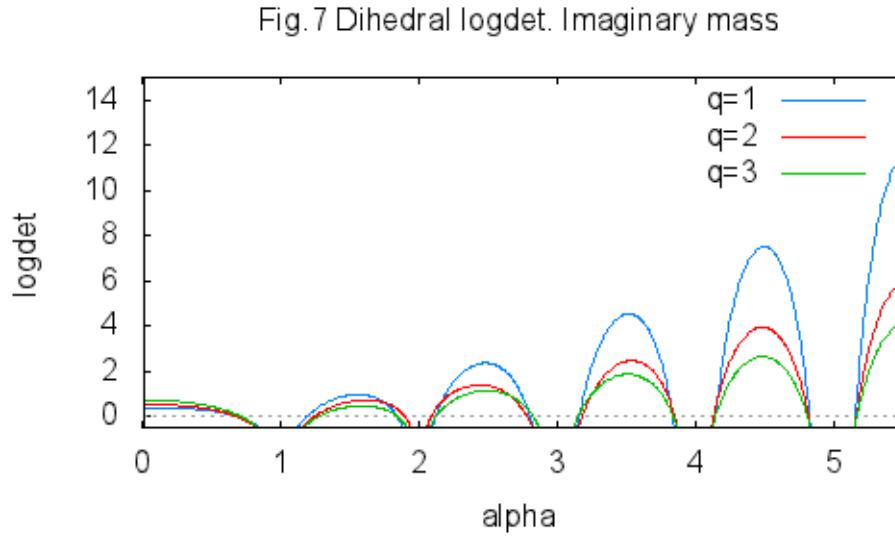
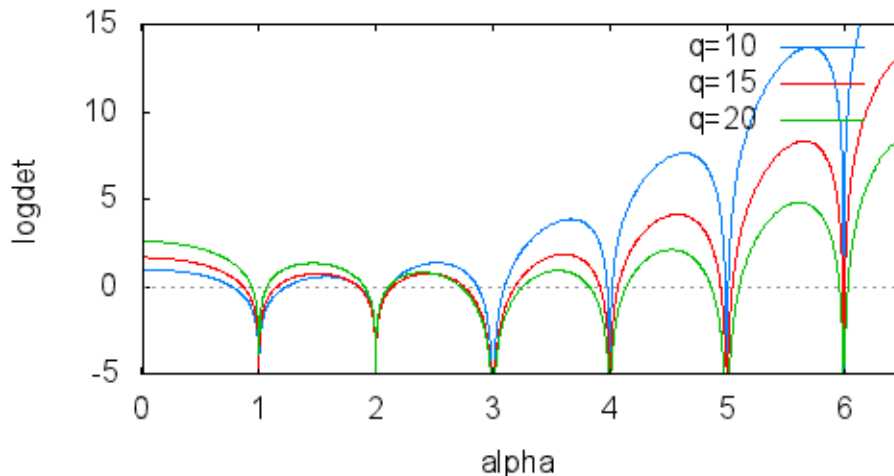
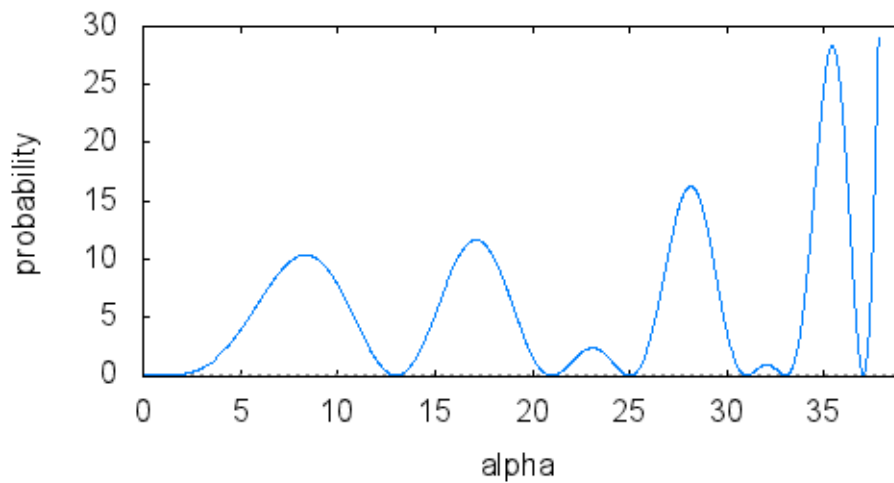


Fig.8 Dihedral logdet. Imaginary mass



Finally, to illustrate the sort of behaviour that occurs, in Figure 9 I display the Hartle–Hawking probability for the $[3, 3, 5]$ case, over a fair range of α . There is actually a shallow zero at $\alpha = 1$, (see Figure 6). I should also note that there is no apparent pattern to the parity of the degeneracies.

Fig.9 $[3,3,5]$ Hartle-Hawking probability



6. Discussion

I have presented more applications of a simple numerical method of computing functional determinants (for negative mass-squared) on spherical quotients, and have shown that one does not have to work too hard to take the more complicated

degeneracies into account. The extension to higher dimensions, is straightforward, if desired.

Concerning the numerical results, I draw no conclusions above and beyond the remark that the, possibly unwelcome [6,8], large- α divergence still seems to be present.

Calculationally, the ω in (14) do not have to take the specific polytope values. They can be any integers, but would then seem to lose geometric significance.

On a purely technical note, I have differentiated twice to get a convergent resolvent. In fact, when calculating the determinant it is possible to get away with differentiating once. This works well for the round sphere, where the degeneracies are simple, and standard summations can be employed. However things are more involved for the higher factors and I could not get this procedure to work very easily, although I have not tried hard in this regard.

It is possible to discuss the case of lunes of *arbitrary* angle but then the generating function (5) cannot be used, at least not to extract the degeneracies..

References.

1. Dowker,J.S. *Imaginary mass lens space determinants*, ArXiv:1404.3495
2. Dowker,J.S., *Massive sphere determinants*, ArXiv:1404:0986.
3. Dowker.J.S. *Entanglement entropy for odd spheres*. ArXiv:1012.1548.
4. Dowker,J.S. *J. Phys.A:Math.Theor.* **46** (2013) 2254; ArXiv:1212.2098
5. Petiau,G. *La Théorie des Fonctions de Bessel*, (CNRS,Paris,1955).
6. Anninos,D., Denef,F. and Harlow,D. *Phys. Rev.* **D88** (2013) 084049.
7. Dowker.J.S. and Kirsten, K. *Analysis and Appl.* **3** (2005) 45; ArXiv:hep-th/0301143.
8. Anninos,D., Denef,F., Konstantinidis,G. and Shaghoulian,E. *Higher Spin de Sitter Holography from Functional Determinants*, ArXiv:1305.6321.
9. Chang,P. and Dowker,J.S. *Nucl. Phys.* **B395** (1993) 407, ArXiv:hep-th/9210013.